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# Do Rosenblatt and Nataf isoprobabilistic transformations really differ?

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ABSTRACT

This article is the third in a series dedicated to the mathematical study of isoprobabilistic transformations and their relationship with stochastic dependence modelling, see [R. Lebrun, A. Dutfoy, An innovating analysis of the Nataf transformation from the viewpoint of copula, Probabilistic Engineering Mechanics (2008). doi: 10.1016/j.probengmech.2008.08.001] for an interpretation of the Nataf transformation in term of normal copula and [R. Lebrun, A. Dutfoy, A generalization of the Nataf transformation to distributions with elliptical copula, Probabilistic Engineering Mechanics (24) (2009), 172–178. doi:10.1016/j.probengmech.2008.05.001] for a generalisation of the Nataf transformation to any elliptical copula.

In this article, we explore the relationship between two isoprobabilistic transformations widely used in the community of reliability analysts, namely the Generalised Nataf transformation and Rosenblatt transformation.

First, we recall the elementary results of the copula theory that are needed in the remaining of the article, as a preliminary section to the presentation of both the Generalized Nataf transformation and the Rosenblatt transformation in the light of the copula theory.

Then, we show that the Rosenblatt transformation using the canonical order of conditioning is identical to the Generalised Nataf transformation in the normal copula case, which is the most usual case in reliability analysis since it corresponds to the classical Nataf transformation. At this step, we also show that it is not possible to extend the Rosenblatt transformation to distributions with general elliptical copula the way the Nataf transformation has been generalised.

Furthermore, we explore the effect of the conditioning order of the Rosenblatt transformation on the usual reliability indicators obtained from a FORM or SORM method. We show that in the normal copula case, all these reliability indicators, excepted the importance factors, are unchanged whatever the conditioning order one chooses.

In the last section, we conclude the article with two numerical applications that illustrate the previous results: the equivalence between both transformations in the normal copula case, and the effect of the conditioning order in the normal and non-normal copula case.



## 1. Introduction

In reliability analysis, the uncertainty related to the input variables is modelled with a multivariate probability distribution that defines a random vector $\underline{X}$ of dimension $n$. We suppose in this article that this distribution is absolutely continuous and thus admit a joint density function. These distributions are propagated through a physical model $f$ to compute the distribution of the output variable $Y$. Stakeholders aim at taking a decision on the basis of the realisation of criteria, which may be the probability that the output variable overpasses a given threshold.

The evaluation of this probability might require a high number of simulations, which might forbid the use of simulation algorithms, particularly if the physical model has a high computational cost.

That is why an alternative to the simulation methods has been developed, based on an isoprobabilistic transformation that maps the physical space into a new space called the *standard space*. In that space, the evaluation of the probability can be done using cheap approximations such as the FORM or SORM approximations.

Two isoprobabilistic transformations are presented in the literature: the Nataf transformation [1,12] which has been extended into a Generalised Nataf transformation [2] and the Rosenblatt transformation [3].

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The main objective of this article is to compare the Generalized Nataf transformation with the Rosenblatt one and to prove that they are identical in the normal copula case, which is the most common case in actual reliability studies as it corresponds to the use of the classical Nataf transformation.

We also study the possibility to modify the Rosenblatt transformation in a similar way that the Nataf transformation has been extended to lead to a non-normal standard space.

The second objective of this article is to study the impact of the conditioning order of the Rosenblatt transformation on the usual reliability indicators obtained after an analytical FORM/SORM method, with a focus on the normal copula case.

We note $F_k^{\underline{X}}$ the $k$-th univariate marginal cumulative distribution function of the random vector $\underline{X}$ and $C^{\underline{X}}$ its copula. We note $F_{1,k}^{\underline{X}}$ the cumulative distribution function of the sub-vector $(X_1, \dots, X_k)$ of $\underline{X}$ and $C_{1,k}^{\underline{X}}$ its copula.
When no confusion is possible, we remove the superscript in order to ease the reading: $F_k^{\underline{X}}$ and $C^{\underline{X}}$ become $F_k$ and $C$.

Furthermore, we note $C_{\underline{\underline{R}}}^N$ is a normal copula whose correlation matrix is $\underline{\underline{R}}$. We suppose that $\underline{\underline{R}}$ is a symmetric positive definite matrix.

We note $\mathbb{M}_{n,n}(\mathbb{R})$ the set of a real square matrix of dimension $n$, $\mathcal{O}_n(\mathbb{R})$ the subset of orthogonal matrices of $\mathbb{M}_{n,n}(\mathbb{R})$, i.e:

$$\forall \underline{\underline{Q}} \in \mathcal{O}_n(\mathbb{R}), \qquad \underline{\underline{Q}}\,\underline{\underline{Q}}^t = \underline{\underline{I}}_n \tag{1}$$

and $\mathcal{GL}_n(\mathbb{R})$ the subset of invertible matrices of $\mathbb{M}_{n,n}(\mathbb{R})$.

If $\underline{\underline{R}} = (r_{ij})_{1\leq i,j\leq n} \in \mathbb{M}_{n,n}(\mathbb{R})$, then $\underline{\underline{R}}_k$ is its $k$-leading sub-block:

$$\underline{\underline{R}}_k = (r_{ij})_{1\leq i,j\leq k} \tag{2}$$

and $\underline{R}^k$ is the $(k+1)$-th partial column vector:

$$\underline{R}^k = (r_{1,k+1}, \dots, r_{k,k+1})^t. \tag{3}$$

We call *standard space* the image space of an isoprobabilistic transformation.

## 2. Elementary results on copula

In this section, we recall some basic results of the theory of copulas. We restrict ourselves to the strict minimum to follow further developments. The interested reader is invited to consult e.g. [4] for a thorough introduction to the theory of copulas, including a demonstration of the results presented here.

The following theorem links the joint cumulative distribution function of a multivariate distribution to a copula:

**Theorem 1** (*Sklar, 1959*). *Let $F$ be a cumulative distribution function of dimension $n$ whose univariate marginal cumulative distribution functions are $F_i$. It exists a copula $C$ of dimension $n$ such that for $\underline{x} \in \overline{\mathbb{R}}^n$, we have:*

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)). \tag{4}$$

*If the marginal distributions $F_i$ are continuous, the copula $C$ is unique; otherwise, it is uniquely determined on* $\text{Range}(F_1) \times \cdots \times \text{Range}(F_n)$.

*In the case of continuous marginal distributions, for all $\underline{u} \in [0,1]^n$, we have:*

$$C(\underline{u}) = F\left(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)\right). \tag{5}$$

The copula is unchanged under almost strictly increasing marginal transformation:

**Proposition 2.** *If $\underline{X}$ has as a copula $C$ and if $(\alpha_1, \dots, \alpha_n)$ are $n$ almost strictly increasing functions defined respectively on the supports of the $X_i$, then $C$ is also the copula of $(\alpha_1(X_1), \dots, \alpha_n(X_n))$.*

If we are interested in the distribution of the sub-vector $(X_1, \dots, X_k)$ of the random vector $\underline{X}$, we have the following property:

**Proposition 3** (*k-dimensional Marginal Distributions*). *Let $\underline{X}$ be a continuous random vector with a distribution defined by its copula $C$ and its marginal cumulative distribution functions $F_i$. The cumulative distribution function $F_{1,k}$ of the $k$-dimensional random vector $(X_1, \dots, X_k)$ is defined by its marginal distributions $F_i$ and the copula $C_{1,k}$ through the relation:*

$$F_{1,k}(x_1, \dots, x_k) = C_{1,k}(F_1(x_1), \dots, F_k(x_k)) \tag{6}$$

*with*

$$C_{1,k}(u_1, \dots, u_k) = C(u_1, \dots, u_k, 1, \dots, 1). \tag{7}$$

**Definition 4** (*k-th Conditional Marginal Distribution*). Let $\underline{X} = (X_1, \dots, X_n)$ be a continuous random vector. The cumulative distribution function of the conditional variable $X_k|X_1, \dots, X_{k-1}$ is defined by:

$$F_{k|1,\dots,k-1}(x_k|x_1, \dots, x_{k-1}) = \frac{\partial^{k-1} F_{1,k}(x_1, \dots, x_k)}{\partial x_1 \cdots \partial x_{k-1}} \Big/ \frac{\partial^{k-1} F_{1,k-1}(x_1, \dots, x_{k-1})}{\partial x_1 \cdots \partial x_{k-1}}. \tag{8}$$

**Proposition 5** (*k-th Conditional Marginal Copula*). *Let $\underline{X}$ be a continuous random vector with a distribution defined by its copula $C$ and its marginal cumulative distribution functions $F_i$. The cumulative distribution function of the conditional variable $X_k|X_1, \dots, X_{k-1}$ is defined by its marginal distributions $F_i$ and the copula $C_{k|1,\dots,k-1}$ through the relation:*

$$F_{k|1,\dots,k-1}(x_k|x_1, \dots, x_{k-1}) = C_{k|1,\dots,k-1}(F_k(x_k)|F_1(x_1), \dots, F_{k-1}(x_{k-1})) \tag{9}$$

*with*

$$C_{k|1,\dots,k-1}(u_k|u_1, \dots, u_{k-1}) = \frac{\partial^{k-1} C_{1,k}(u_1, \dots, u_k)}{\partial u_1 \cdots \partial u_{k-1}} \Big/ \frac{\partial^{k-1} C_{1,k-1}(u_1, \dots, u_{k-1})}{\partial u_1 \cdots \partial u_{k-1}}. \tag{10}$$

As a matter of fact, relation (10) is the direct application of Definition 4 to the cumulative distribution $C$. Furthermore, Definition 4 and relation (4) lead to:

$$F_{k|1,\dots,k-1}(x_k|x_1, \dots, x_{k-1}) = \left[\prod_{i=1}^{i=k-1} p_i(x_i)\right] \frac{\partial^{k-1} C_{1,k}(F_1(x_1), \dots, F_k(x_k))}{\partial u_1 \cdots \partial u_{k-1}} \Big/ \cdots \left[\prod_{i=1}^{i=k-1} p_i(x_i)\right] \frac{\partial^{k-1} C_{1,k-1}(F_1(x_1), \dots, F_k(x_{k-1}))}{\partial u_1 \cdots \partial u_{k-1}} \tag{11}$$

where $p_i$ is the probability density function of $X_i$.

Relations (11) and (10) lead to relation (9).

## 3. The Generalised Nataf and Rosenblatt transformations

The Nataf transformation [1] has been introduced as a procedure to transform the univariate marginal distributions of a multivariate marginal distribution. Its usage in reliability analysis has been popularised by several authors, see e.g. [5,6]. By the way, the Nataf transformation has been presented as a way to perform reliability computations using incomplete dependence information, and it is only recently that its role as a probabilistic modelling tool has been emphasised, linked to the copula theory [7]: the traditional usage of the Nataf transformation supposes that the random vector $\underline{X}$ has a normal copula. A generalisation to random vectors with any elliptical copulas has been proposed in [2], and this is the generalisation which is recalled here:

**Definition 6** (*Generalised Nataf Transformation*). Let $\underline{X}$ in $\mathbb{R}^n$ be a continuous random vector defined by its univariate marginal cumulative distribution functions $F_i^{\underline{X}}$ and its elliptical copula $C^{\underline{X}}$, characterized by its generator $\psi$. The *Generalised Nataf transformation* $T^{GN}$ is defined by:

$$\underline{U} = T^{GN}(\underline{X}) = T_2^{GN} \circ T_1^{GN}(\underline{X}) \tag{12}$$

where both transformations $T_1^{GN}$ and $T_2^{GN}$ are given by:

$$\begin{aligned} T_1^{GN} : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ \underline{X} &\mapsto \underline{W} = \begin{pmatrix} E^{-1} \circ F_1^{\underline{X}}(X_1) \\ \cdots \\ E^{-1} \circ F_n^{\underline{X}}(X_n) \end{pmatrix} \\ T_2^{GN} : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ \underline{W} &\mapsto \underline{U} = \underline{\underline{\Gamma}}\, \underline{W} \end{aligned} \tag{13}$$

where $E$ is the cumulative distribution function of univariate standard elliptical distribution with characteristic generator $\psi$ and $\underline{\underline{\Gamma}}$ is the inverse of the Cholesky factor of $\underline{\underline{R}}$.

Another widely used isoprobabilistic transformation is the Rosenblatt transformation [3], defined by:

**Definition 7** (*Rosenblatt Transformation*). Let $\underline{X}$ in $\mathbb{R}^n$ be a continuous random vector defined by its univariate marginal cumulative distribution functions $F_i^{\underline{X}}$ and its copula $C^{\underline{X}}$. The *Rosenblatt transformation* $T^R$ of $\underline{X}$ is defined by:

$$\underline{U} = T^R(\underline{X}) = T_2^R \circ T_1^R(\underline{X}) \tag{14}$$

where both transformations $T_1^R$, and $T_2^R$ are given by:

$$\begin{aligned} T_1^R : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ \underline{X} &\mapsto \underline{Y} = \begin{pmatrix} F_1^{\underline{X}}(X_1) \\ \vdots \\ F_{k|1,\ldots,k-1}^{\underline{X}}(X_k|X_1,\ldots,X_{k-1}) \\ \vdots \\ F_{n|1,\ldots,n-1}^{\underline{X}}(X_n|X_1,\ldots,X_{n-1}) \end{pmatrix} \\ T_2^R : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ \underline{Y} &\mapsto \underline{U} = \begin{pmatrix} \Phi^{-1}(Y_1) \\ \vdots \\ \Phi^{-1}(Y_n) \end{pmatrix} \end{aligned} \tag{15}$$

where $F_{k|1,\ldots,k-1}^{\underline{X}}$ is the cumulative distribution function of the conditional random variable $X_k|X_1,\ldots,X_{k-1}$.

Let us note that $T_1^R$ maps $\underline{X}$ into a uniformly distributed random vector over $[0,1]^n$ with independent copula. For the demonstration, see [5, chap.7̃.2, p. 123].

Furthermore, $T_2^R$ maps $\underline{Y}$ into a normally distributed random vector with zero mean and unit covariance matrix: as a mater of fact, Proposition 2 implies that $\underline{U}$ has the same copula as $\underline{Y}$. Besides, by construction, the univariate marginal distributions of $\underline{U}$ are standard normal.

In order to ease the further comparison between the generalised Nataf transformation and the Rosenblatt one, it is useful to rewrite the Rosenblatt transformation as follows:

**Definition 8** (*Rosenblatt Transformation, New Formulation*). Let $\underline{X}$ in $\mathbb{R}^n$ be a continuous random vector defined by its univariate marginal cumulative distribution functions $F_i^{\underline{X}}$ and its copula $C^{\underline{X}}$. The *new formulation of the Rosenblatt transformation* $T^{NR}$ is defined by:

$$\underline{U} = T^{NR}(\underline{X}) = T^R \circ T_0(\underline{X}) \tag{16}$$

where $T_0$ is given by:

$$\begin{aligned} T_0 : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ \underline{X} &\mapsto \underline{W} = \begin{pmatrix} \Phi^{-1} \circ F_1^{\underline{X}}(X_1) \\ \cdots \\ \Phi^{-1} \circ F_n^{\underline{X}}(X_n) \end{pmatrix} \end{aligned} \tag{17}$$

where $\Phi$ is the cumulative distribution function of the univariate standard normal distribution, $T^R$ the Rosenblatt transformation of Definition 7.

Let us note that $\underline{U}^{NR} = T^{NR}(\underline{X}) = T_2^R \circ T_1^R \circ T_0(\underline{X})$.

If $\underline{W} = T_0(\underline{X})$, then, thanks to (15), the $k^{th}$ component of $\underline{U}^{NR}$ writes:

$$U_k^{NR} = \Phi^{-1} \circ F_{k|1,\ldots,k-1}^{\underline{W}}(W_k|W_1,\ldots,W_{k-1}). \tag{18}$$

Thanks to Proposition 5, the cumulative distribution function of the conditional variable $W_k|W_1,\ldots,W_{k-1}$ writes:

$$\begin{aligned} &F_{k|1,\ldots,k-1}^{\underline{W}}(w_k|w_1,\ldots,w_{k-1}) \\ &= C_{k|1,\ldots,k-1}^{\underline{W}}\left(F_k^{\underline{W}}(w_k)|F_1^{\underline{W}}(w_1),\ldots,F_{k-1}^{\underline{W}}(w_{k-1})\right). \end{aligned} \tag{19}$$

From Proposition 2, it follows that $\underline{X}$ and $\underline{W}$ have the same copula $C^{\underline{X}}$. Furthermore, by construction of $\underline{W}$, we have $F_k^{\underline{W}} = \Phi$ and $\Phi(W_k) = F_k^{\underline{X}}(X_k)$. Then, relation (19) rewrites:

$$\begin{aligned} &F_{k|1,\ldots,k-1}^{\underline{W}}(W_k|W_1,\ldots,W_{k-1}) \\ &= C_{k|1,\ldots,k-1}^{\underline{X}}\left(F_k^{\underline{X}}(X_k)|F_1^{\underline{X}}(X_1),\ldots,F_{k-1}^{\underline{X}}(X_{k-1})\right) \end{aligned} \tag{20}$$

which finally leads to the relation:

$$F_{k|1,\ldots,k-1}^{\underline{W}}(W_k|W_1,\ldots,W_{k-1}) = F_{k|1,\ldots,k-1}^{\underline{X}}(X_k|X_1,\ldots,X_{k-1}) \tag{21}$$

and then to:

$$U_k^{NR} = \Phi^{-1} \circ F_{k|1,\ldots,k-1}^{\underline{X}}(X_k|X_1,\ldots,X_{k-1}) \tag{22}$$

which is precisely the expression of the Rosenblatt transformation of Definition 7:

$\underline{U}^{NR} = T^R(\underline{X})$.

## 4. Do generalised Nataf and Rosenblatt transformations really differ?

In this section, we first consider the case where the copula of $\underline{X}$ is normal, which is the most usual case in reliability analysis since it corresponds to the case where the classical Nataf transformation applies.
Then, we make the comparison in all the other cases: non-normal elliptical copulas and non-elliptical copulas.

### 4.1. The normal copula case

The new formulation (16) of the Rosenblatt transformation makes it easier to show that when $\underline{X}$ has a normal copula, both transformations are identical:

**Proposition 9** (*Isoprobabilistic Transformations, Normal Copula*). *Let $\underline{X}$ in $\mathbb{R}^n$ be a continuous random vector defined by its univariate marginal cumulative distribution functions $F_i^{\underline{X}}$ and its normal copula $C_{\underline{\underline{R}}}^N$. Then, the Rosenblatt transformation and the generalised Nataf one are identical:*

$$T^R(\underline{X}) = T^{GN}(\underline{X}). \tag{23}$$

We recall without demonstration some well-known results about normal vectors and an easy-to demonstrate result on orthogonal matrices that will be used in the demonstration of Proposition 9.

**Proposition 10** (*Conditional Normal Vector*). *Let $\underline{U} = (\underline{X}_1, \underline{X}_2)^t$ in $\mathbb{R}^{n_1+n_2}$ be a normal random vector. Then the conditional random vector $\underline{V} = \underline{X}_2|\underline{X}_1$ is a normal vector which mean vector and covariance matrix are defined by:*

$$\begin{cases} \mathrm{E}[\underline{V}] = \mathrm{E}[\underline{X}_2] + \underline{\underline{\mathrm{Cov}}}(\underline{X}_2, \underline{X}_1)[\underline{\underline{\mathrm{Cov}}}(\underline{X}_1, \underline{X}_1)]^{-1}(\underline{X}_1 - \mathrm{E}[\underline{X}_1]) \\ \underline{\underline{\mathrm{Cov}}}(\underline{V}) = \underline{\underline{\mathrm{Cov}}}(\underline{X}_2, \underline{X}_2) - \underline{\underline{\mathrm{Cov}}}(\underline{X}_2, \underline{X}_1)[\underline{\underline{\mathrm{Cov}}}(\underline{X}_1, \underline{X}_1)]^{-1}\underline{\underline{\mathrm{Cov}}}(\underline{X}_1, \underline{X}_2). \end{cases} \tag{24}$$

**Proposition 11** (*Affine Transformation*). *Let $\underline{X}$ in $\mathbb{R}^n$ be a normal vector, with mean vector is $\underline{\mu}$, and covariance matrix $\underline{\underline{\Sigma}}$, $\underline{\underline{A}}$ a deterministic matrix in $\mathbb{M}_{n,p}(\mathbb{R})$ and $\underline{b}$ in $\mathbb{R}^p$ a deterministic vector. Then $\underline{Y} = \underline{\underline{A}}\,\underline{X} + \underline{b}$ is a normal vector whose mean vector and covariance matrix are defined by:*

$$\begin{cases} \mathrm{E}[\underline{Y}] = \underline{\underline{A}}\,\underline{\mu} + \underline{b} \\ \underline{\underline{\mathrm{Cov}}}(\underline{Y}) = \underline{\underline{A}}\,\underline{\underline{\Sigma}}\,\underline{\underline{A}}^t. \end{cases} \tag{25}$$

**Proposition 12** (*Orthogonal Triangular Matrix with Positive Diagonal*). *Let $\mathcal{T}^+(\mathbb{R})$ be the set of lower triangular matrix of $\mathbb{M}_{n,n}(\mathbb{R})$ with positive diagonal elements. Then $\mathcal{T}^+(\mathbb{R})$ is a multiplicative subgroup of $\mathcal{GL}_n(\mathbb{R})$.*
*Furthermore, $\mathcal{T}^+(\mathbb{R}) \cap \mathcal{O}_n(\mathbb{R}) = \{\underline{\underline{I}}_n\}$.*

We can now start to demonstrate Proposition 9, using the new formulation of the Rosenblatt transformation of Definition 8, whose different steps are the following ones:

$$T^{NR}: \underline{X} \xmapsto{T_0} \underline{W} \xmapsto{T_1^R} \underline{Y} \xmapsto{T_2^R} \underline{U}. \tag{26}$$

Let us note $\underline{S}_{k-1} = (W_1, \dots, W_{k-1})^t$ and $V_k = W_k|\underline{S}_{k-1}$.
As $\underline{X}$ has a normal copula, $\underline{W}$ is a $n$-dimensional normal vector whose univariate marginal distributions are standard normal and which correlation matrix is $\underline{\underline{R}}$.

Proposition 10 gives that for all $k$, $V_k$ follows a univariate normal distribution and relation (24) leads to:

$$\begin{aligned} \mathrm{E}[V_k] &= \mathrm{E}[W_k] + \underline{\underline{\mathrm{Cov}}}(W_k, \underline{S}_{k-1})[\underline{\underline{\mathrm{Cov}}}(\underline{S}_{k-1}, \underline{S}_{k-1})]^{-1}(\underline{S}_{k-1} - \mathrm{E}[\underline{S}_{k-1}]) \\ &= \underline{\underline{\mathrm{Cov}}}(W_k, \underline{S}_{k-1})[\underline{\underline{\mathrm{Cov}}}(\underline{S}_{k-1}, \underline{S}_{k-1})]^{-1}\underline{S}_{k-1}. \end{aligned} \tag{27}$$

Besides, we have $\underline{\underline{\mathrm{Cov}}}(\underline{W}) = \underline{\underline{\mathrm{Cor}}}(\underline{W}) = \underline{\underline{R}}$, and given the notations (2) and (3), we have:

$$\mathrm{E}[V_k] = \left(\underline{R}^{k-1}\right)^t [\underline{\underline{R}}_{k-1}]^{-1}\underline{S}_{k-1}. \tag{28}$$

Furthermore, relation (24) also leads to:

$$\begin{aligned} \mathrm{Var}[V_k] &= \mathrm{Var}(W_k) - \underline{\underline{\mathrm{Cov}}}(W_k, \underline{S}_{k-1})[\underline{\underline{\mathrm{Cov}}}(\underline{S}_{k-1}, \underline{S}_{k-1})]^{-1} \\ &\quad \times \underline{\underline{\mathrm{Cov}}}(\underline{S}_{k-1}, W_k) \\ &= 1 - \left(\underline{R}^{k-1}\right)^t [\underline{\underline{R}}_{k-1}]^{-1}\underline{R}^{k-1}. \end{aligned} \tag{29}$$

Given relations (28) and (29), the $k^{th}$ component of $\underline{Y}$ is defined by:

$$\begin{aligned} Y_k &= F^{\underline{W}}_{k|1,\dots,k-1}(W_k|W_1, \dots, W_{k-1}) \\ &= \Phi\left(\frac{W_k - \left(\underline{R}^{k-1}\right)^t [\underline{\underline{R}}_{k-1}]^{-1}\underline{S}_{k-1}}{\sqrt{1 - \left(\underline{R}^{k-1}\right)^t [\underline{\underline{R}}_{k-1}]^{-1}\underline{R}^{k-1}}}\right). \end{aligned} \tag{30}$$

Finally, we obtain:

$$U_k = \Phi^{-1}(Y_k) = \frac{W_k - \left(\underline{R}^{k-1}\right)^t [\underline{\underline{R}}_{k-1}]^{-1}\underline{S}_{k-1}}{\sqrt{1 - \left(\underline{R}^{k-1}\right)^t [\underline{\underline{R}}_{k-1}]^{-1}\underline{R}^{k-1}}} = \underline{\underline{A}}_k\underline{W} \tag{31}$$

where for all $k \in [1, n]$, $\underline{\underline{A}}_k = (a_{k,1}, \dots, a_{k,k}, 0, \dots, 0) \in \mathcal{M}_{1n}(\mathbb{R})$ with:

$$\begin{cases} a_{k,k} = \left[\sqrt{1 - \left(\underline{R}^{k-1}\right)^t [\underline{\underline{R}}_{k-1}]^{-1}\underline{R}^{k-1}}\right]^{-1} \\ a_{k,j} = -a_{k,k}\displaystyle\sum_{i=1}^{i=k-1} r_{1i}r_{ji} \quad \text{for } \forall j \in [1, k-1]. \end{cases} \tag{32}$$

As $\underline{\underline{A}}_k$ is a row matrix, $U_k$ only depends on $\underline{S}_k$. Let $\underline{\underline{\tilde{\Gamma}}}$ be the lower triangular matrix which line $k$ is $\underline{\underline{A}}_k$. Then relation (31) implies that:

$$\underline{U} = \underline{\underline{\tilde{\Gamma}}}\,\underline{W} \tag{33}$$

which is very close to relation (13). It remains to show that $\underline{\underline{\tilde{\Gamma}}} = \underline{\underline{\Gamma}}$.

Proposition 11 implies that $\underline{\underline{\mathrm{Cov}}}(\underline{U}) = \underline{\underline{\tilde{\Gamma}}}\,\underline{\underline{R}}\,\underline{\underline{\tilde{\Gamma}}}^t$ and $\underline{\underline{\mathrm{Cov}}}(\underline{U}) = \underline{\underline{I}}_n$ by construction of $\underline{U}$. If $\underline{\underline{L}}$ is the Cholesky factor of $\underline{\underline{R}}$, then $\underline{\underline{R}} = \underline{\underline{L}}\,\underline{\underline{L}}^t$, and $(\underline{\underline{\tilde{\Gamma}}}\,\underline{\underline{L}})(\underline{\underline{\tilde{\Gamma}}}\,\underline{\underline{L}})^t = \underline{\underline{I}}_n$, which leads to $\underline{\underline{\tilde{\Gamma}}}\,\underline{\underline{L}} \in \mathcal{O}_n(\mathbb{R})$. Furthermore, by construction, $\underline{\underline{\tilde{\Gamma}}} \in \mathcal{T}^{+*}(\mathbb{R})$. As $\underline{\underline{L}} \in \mathcal{T}^{+*}(\mathbb{R})$, Proposition 12 implies that $\underline{\underline{\tilde{\Gamma}}}\,\underline{\underline{L}} \in \mathcal{T}^{+*}(\mathbb{R})$ and $\underline{\underline{\tilde{\Gamma}}}\,\underline{\underline{L}} = \underline{\underline{I}}_n$, which rewrites $\underline{\underline{\tilde{\Gamma}}} = \underline{\underline{L}}^{-1} = \underline{\underline{\Gamma}}$.

In conclusion, we showed that in the case where $\underline{X}$ has a normal copula, we have the relation $T_2^R \circ T_1^R \circ T_0(\underline{X}) = T_2^N \circ T_1^N(\underline{X})$ which leads to:

$$T^R(\underline{X}) = T^N(\underline{X}). \tag{34}$$

Thus, the equivalence of the Rosenblatt transformation and the Generalised Nataf transformation in the normal copula case is shown.

### 4.2. The other cases

In the case where the copula of $\underline{X}$ is elliptical but non-normal, both isoprobabilistic transformations differ as their associated standard spaces are different. As a matter of fact, the standard spaces of the Generalised Nataf is associated with the standard spherical representative of the elliptical family that defines the elliptical copula, whereas the standard space of the Rosenblatt transformation is associated to the normal distribution.

At this step, it is interesting to check whether it is possible to modify the Rosenblatt transformation in order to make its standard space be the same as the one associated with the generalised Nataf transformation.

In [7] we recall that the essential characteristic of the standard space is the spherical symmetry of its associated distribution, which gives a sense to the FORM and SORM approximations of the event probability.
Let us note that by construction, because of the conditioning step $T_1^R$, the Rosenblatt transformation leads to a final vector $\underline{U}$ with an independent copula.

**Fig. 1.** Rosenblatt transformations when the conditioning of the components $W_k$ follow the canonical order or an arbitrary order.

**Proposition 13** (*Spherical Distribution with Independent Copula*). *The only spherical distributions with independent components are the normal distributions with zero mean and scalar covariance matrix $\lambda \underline{\underline{I}}_n$ with $\lambda > 0$.*

See [8] for a demonstration.

Thus, the only way to map a random vector with an independent copula into a random vector following a spherical distribution, is to map it into a normal vector such as described in this proposition: thus, the standard space of the Rosenblatt transformation is necessarily the normal one.

Therefore, the standard space of the Rosenblatt transformation and the standard space of the Generalised Nataf transformation only coincide in the normal copula case.

At last, for all the other cases where the copula of $\underline{X}$ is not elliptical, the Generalised Nataf transformation is not defined and the comparison with the Rosenblatt transformation not possible.

## 5. Impact of the conditioning order in the Rosenblatt transformation in the normal copula case

In the literature [5], the presentation of the Rosenblatt transformation is given with the warning that the conditioning order in step $T_1^R$ has an impact on the results obtained from a FORM/SORM method.

Let us call *canonical order* the order presented in the relation (15).

In that section, we study the impact of a change in the conditioning order of the Rosenblatt transformation on the quantities evaluated in the context of the use of the FORM or SORM methods: the design point, which is used through its norm (reliability index) and its normalised squared coordinates (importance factors), and the curvatures of the limit state surface at the design point in the standard space, where the limit state surface is the frontier of the subspace of parameters verifying the event (for SORM approximation).

In the case where the copula of $\underline{X}$ is not normal, it has already been shown that such a change has an impact on all these elements: see the example quoted by [9] and discussed by several authors, for example [5] and [10].

However, this is not always the case. We will study in more detail the most frequent situation where the copula of $\underline{X}$ is normal since, as mentioned previously, it is the copula induced by the use of the classical Nataf transformation.

Let us suppose now that we change the order of conditioning. It is equivalent to consider the introduction of a new step in the Rosenblatt transformation between the steps $T_0$ and $T_1^R$ of relation (26) in order to make a permutation of the components of $\underline{W}$. We note $\underline{\underline{P}}$ the permutation matrix such as $\underline{W}_2 = \underline{\underline{P}}\,\underline{W}$.

The Rosenblatt transformations using the canonical order or an arbitrary order are summarised graphically in Fig. 1.

We have the following result:

**Proposition 14** (*Impact of the Order of Conditioning, Normal Copula*). *In the normal copula case, changing the order of the conditioning in the Rosenblatt transformation consists in making an orthogonal transformation in the standard space of the Rosenblatt transformation.*

*More precisely, if we note $\underline{\underline{P}}$ the permutation matrix associated to the arbitrary order, $T_2^R$ the Rosenblatt transformation associated to this ordering, $\underline{U}_2 = T_2^R(\underline{X})$ and $\underline{U} = T^R(\underline{X})$, then we have:*

$$\exists \underline{\underline{Q}} \in \mathcal{O}_n(\mathbb{R}) \,/\, \underline{U}_2 = \underline{\underline{Q}}\,\underline{U} \tag{35}$$

*where $\underline{\underline{Q}}$ and $\underline{\underline{P}}$ are in the same connected component of $\mathcal{O}_n(\mathbb{R})$, it means $\det \underline{\underline{P}} = \det \underline{\underline{Q}}$.*

*According to the notations of Fig. 1, if $\underline{\underline{R}}$ is the correlation matrix of the normal copula of $\underline{X}$, $\underline{\underline{R}}_2$ the one of $\underline{W}_2$, $\underline{\underline{\Gamma}}$ and $\underline{\underline{\Gamma}}_2$ the inverse of their respective Cholesky factors, then the matrices $\underline{\underline{P}}$ and $\underline{\underline{Q}}$ are linked by:*

$$\underline{\underline{Q}} = \underline{\underline{\Gamma}}_2\,\underline{\underline{P}}\,\underline{\underline{\Gamma}}^{-1}. \tag{36}$$

The following result will help for the demonstration of Proposition 14:

**Proposition 15** (*Triangular Decomposition*). *Let $\underline{\underline{A}}$ and $\underline{\underline{B}}$ be two deterministic matrices in $\mathbb{M}_{n,n}(\mathbb{R})$, with $\underline{\underline{B}}$ invertible. Then we have:*

$$\underline{\underline{A}}\,\underline{\underline{A}}^t = \underline{\underline{B}}\,\underline{\underline{B}}^t \Longrightarrow \underline{\underline{B}}^{-1}\underline{\underline{A}} \in \mathcal{O}_n(\mathbb{R}) \tag{37}$$

*which means that $\exists \underline{\underline{Q}} \in \mathcal{O}_n(\mathbb{R}) \,/\, \underline{\underline{A}} = \underline{\underline{B}}\,\underline{\underline{Q}}$.*

As a matter of fact, we have the following implications:

$$(\underline{\underline{B}}^{-1}\underline{\underline{A}})(\underline{\underline{B}}^{-1}\underline{\underline{A}})^t = \underline{\underline{B}}^{-1}\underline{\underline{A}}\,\underline{\underline{A}}^t\,\underline{\underline{B}}^{-t} = \underline{\underline{B}}^{-1}\underline{\underline{B}}\,\underline{\underline{B}}^t\,\underline{\underline{B}}^{-t} = \underline{\underline{I}}_n \tag{38}$$

which leads to the result of Proposition 15.

As $\underline{W}_2 = \underline{\underline{P}}\,\underline{W}$, $\underline{W}_2$ is a normal vector which correlation matrix verifies $\underline{\underline{R}}_2 = \underline{\underline{P}}\,\underline{\underline{R}}\,\underline{\underline{P}}^t$ and which Cholesky factor is $\underline{\underline{L}}_2 = \underline{\underline{\Gamma}}_2^{-1}$. Therefore, $\underline{\underline{R}}_2 = \underline{\underline{L}}_2\,\underline{\underline{L}}_2^t = (\underline{\underline{P}}\,\underline{\underline{L}})(\underline{\underline{P}}\,\underline{\underline{L}})^t$. Proposition 15 leads to:

$$\exists \underline{\underline{Q}} \in \mathcal{O}_n(\mathbb{R}) \,/\, \underline{\underline{P}}\,\underline{\underline{L}} = \underline{\underline{L}}_2\,\underline{\underline{Q}}. \tag{39}$$

By multiplying the relation (39) on the left by $\underline{\underline{\Gamma}}_2$ and on the right by $\underline{\underline{\Gamma}}$, it rewrites:

$$\underline{\underline{\Gamma}}_2\,\underline{\underline{P}} = \underline{\underline{Q}}\,\underline{\underline{\Gamma}} \tag{40}$$

which leads to the relation between $\underline{\underline{P}}$ and $\underline{\underline{Q}}$ given in relation (36).

We showed that in the normal copula case, the mapping from $\underline{W}_2$ into $\underline{U}_2$ is linear such as: $\underline{U}_2 = \underline{\underline{\Gamma}}_2\,\underline{W}_2$. Finally, we obtain:

$$\underline{U}_2 = \underline{\underline{\Gamma}}_2\,\underline{\underline{P}}\,\underline{W}. \tag{41}$$

Relations (40) and (41) finally imply that:

$$\underline{U}_2 = \underline{\underline{Q}}\,\underline{\underline{\Gamma}}\,\underline{W} = \underline{\underline{Q}}\,\underline{U} \tag{42}$$

as required.

Given that $\det(\underline{\underline{\Gamma}}) > 0$ and $\det(\underline{\underline{\Gamma}}_2) > 0$, relation (36) implies that $\det(\underline{\underline{Q}})$ and $\det(\underline{\underline{P}})$ have the same sign, which signifies that they belong to the same connected component of $\mathcal{O}_n(\mathbb{R})$.

In conclusion, if the random vector $\underline{X}$ has a normal copula, the effect of changing the order of conditioning in the Rosenblatt transformation with respect to the canonical order is to apply a further orthogonal transformation after applying the Rosenblatt transformation associated to the canonical ordering. It changes the location of the design point, therefore its coordinates, but neither its norm nor the curvatures of the limit state surface at the design point.

Thus, in the context of the FORM or SORM method, the following quantities do not depend on the conditioning order of the Rosenblatt transformation:

- the Hasofer reliability index [11], which is the norm of the design point,
- the FORM approximation of the event probability which relies only on the Hasofer reliability index,
- the SORM approximations of the event probability which rely on both the Hasofer reliability index and the curvatures of the limit state function at the design point.

However, the importance factors which are evaluated from the normalised squared coordinates of the design point change in a way which is not in general a permutation of the values obtained using the canonical order: relation (36) implies that in general, $\underline{\underline{Q}} \neq \underline{\underline{P}}$.
To be more precise, let us consider the following proposition:

**Proposition 16** (*Ordering Indifference*). ($\forall$ *permutation matrix* $\underline{\underline{P}}$, $\underline{\underline{Q}} = \underline{\underline{P}}$) $\Longleftrightarrow$ ($\underline{X}$ *has an independent copula.*)

The first implication is obvious: if $\underline{X}$ has an independent copula, the correlation matrix $\underline{\underline{R}}$ is equal to the identity matrix $\underline{\underline{I}}_n$, which implies that $\underline{\underline{R}}_2 = \underline{\underline{I}}_n$, $\underline{\underline{\Gamma}} = \underline{\underline{I}}_n$, $\underline{\underline{\Gamma}}_2 = \underline{\underline{I}}_n$ and finally $\underline{\underline{Q}} = \underline{\underline{P}}$.
The second implication derives from the following computation. By definition of $\underline{\underline{\Gamma}}_2$ and $\underline{\underline{\Gamma}}$, we have:

$$\underline{\underline{Q}} = \underline{\underline{P}} \Longrightarrow \underline{\underline{L}}_2 = \underline{\underline{P}}\,\underline{\underline{L}}\,\underline{\underline{P}}^t \tag{43}$$

which implies the following relation on the coefficients of $\underline{\underline{L}}_2 = (l^2_{i,j})_{1\leq i,j\leq n}$ and $\underline{\underline{L}} = (l_{i,j})_{1\leq i,j\leq n}$:

$$l^2_{i,j} = l_{\sigma(i),\sigma(j)} \tag{44}$$

where $\sigma$ is the permutation associated to $\underline{\underline{P}}$.
Thus, given that $\underline{\underline{L}}$ and $\underline{\underline{L}}_2$ are lower triangular matrices, if the relation (44) must hold for all the permutations $\sigma$, it must hold in particular for any transposition $\tau_{ij}$ that exchanges $i$ and $j$, thus if $i < j$, $l^2_{i,j} = 0$ by construction, thus $l_{\sigma(i),\sigma(j)} = l_{ji} = 0$: $\underline{\underline{L}}$ is a diagonal matrix and consequently, $\underline{\underline{R}} = \underline{\underline{I}}$, which is equivalent to the independence of the components of $\underline{X}$ in the normal copula case.

In conclusion, a permutation with respect to the canonical order on the components of $\underline{X}$ always corresponds to the same permutation with respect to the canonical order of the components of the standard space random vector only on the independent case. Otherwise, the choice of the conditioning order does not translate into a simple permutation of the values of the design point coordinates. This remark re-enforces the fact that it is difficult to interpret the importance factors on a component basis in the case of correlated variables.

**Remark 17.** A similar question may be raised concerning the Nataf transformation, namely the choice of $\underline{\underline{\Gamma}}$, which is by no way restricted to the Cholesky factor for the standard space to be associated with a spherical distribution: any square root of the correlation matrix would suit. In particular, if $\underline{\underline{P}}$ is a permutation matrix (and more generally any orthogonal matrix), $\underline{\underline{L}}\,\underline{\underline{P}}$ is such a square root.

Let us recall that the exact value of the event probability remains unchanged whatever the transformation we use, and whatever the conditioning order we use for the Rosenblatt transformation!

## 6. Numerical applications

In this section, we illustrate the results obtained in the previous sections through two numerical applications.
We consider a bi-dimensional random vector $\underline{X} = (X_1, X_2)$ defined by its marginal cumulative distribution functions $(F_1, F_2)$ and its copula $C$.
For both applications, we choose exponential distributions $X_1 \sim \mathcal{E}xp(\lambda_1)$ and $X_2 \sim \mathcal{E}xp(\lambda_2)$ for the marginal distributions and a limit state surface defined by:

$$8X_1 + 2X_2 - 1 = 0. \tag{45}$$

We consider the event:

$$8X_1 + 2X_2 - 1 \leq 0 \tag{46}$$

which we want to evaluate the probability.

In the first application, we choose a normal copula $C^N_{\underline{\underline{R}}}$ where $\underline{\underline{R}} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ and $\rho$ the correlation coefficient.
We check both the equivalence between the canonical Rosenblatt transformation and the Generalised Nataf transformation, and the effect of a change in the conditioning order.

In the second application, we choose non-elliptical copula, namely the Frank copula $C_\theta$, which belongs to the class of archimedean copulas, and we verify that a change in the conditioning order is not equivalent to an orthogonal modification of the transformation and has an impact on the FORM and SORM approximations.

We recall that the Frank copula is defined on $[0, 1]^2$ by the expression:

$$C_\theta(u_1, u_2) = -\frac{1}{\theta} \log\left(1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}\right) \tag{47}$$

where $\theta \in \mathbb{R}^*$. For $\theta = 0$, $C_\theta$ is the independent copula.

In the numerical applications, we take $\lambda_1 = 1$, $\lambda_2 = 3$, $\rho = 1/2$ and $\theta = 10$.

### 6.1. Application 1: Normal copula

We use the new expression of the Rosenblatt transformation of Definition 8, with the previous notation $\underline{W} = T_0(\underline{X})$, in that particular case of normal copula.
Given Proposition 10, $W_2|W_1$ is a normal random vector such as $E[W_2|W_1] = \rho W_1$ and $\mathrm{Var}[W_2|W_1] = 1 - \rho^2$, which implies that $F^{W_2|W_1}(W_2|W_1) = \Phi\left(\frac{W_2 - \rho W_1}{\sqrt{1-\rho^2}}\right)$.
Finally, the random vector $\underline{U}$ is defined by:

$$\begin{cases} U_1 = \Phi^{-1} \circ F^{W_1}(W_1) = W_1 \\ U_2 = \Phi^{-1} \circ F^{W_2|W_1}(W_2|W_1) = \dfrac{W_2 - \rho W_1}{\sqrt{1-\rho^2}}. \end{cases} \tag{48}$$

The Rosenblatt transformation with canonical order on the conditioning step finally defines the normal random vector $\underline{U}$ as:

$$\begin{cases} U_1 = \Phi^{-1} \circ F^1(X_1) \\ U_2 = \dfrac{\Phi^{-1} \circ F^2(X_2) - \rho\Phi^{-1} \circ F^1(X_1)}{\sqrt{1-\rho^2}}. \end{cases} \tag{49}$$

In the Rosenblatt standard space, the limit state surface has the parametric expression, where $\xi \in [0, +\infty[$:

$$\begin{cases} u_1 = \Phi^{-1} \circ F^1(\xi) \\ u_2 = \dfrac{\Phi^{-1} \circ F^2\left(\frac{1-8\xi}{2}\right) - \rho\Phi^{-1} \circ F^1(\xi)}{\sqrt{1-\rho^2}}. \end{cases} \tag{50}$$

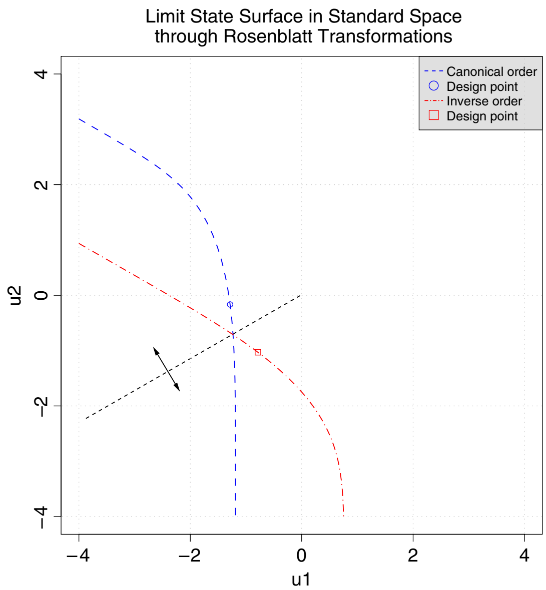


**Fig. 2.** Transformations of the limit state surface into the standard space when using the canonical order in the Rosenblatt transformation and its inverse. The linear correlation is $\rho = 1/2$, the copula is normal, $X_1 \sim \mathcal{E}xp(1)$ and $X_2 \sim \mathcal{E}xp(3)$. The limit state surface is $8X_1 + 2X_2 - 1 = 0$. Note the symmetry that exchanges the two curves: its matrix is $\underline{\underline{Q}}$.

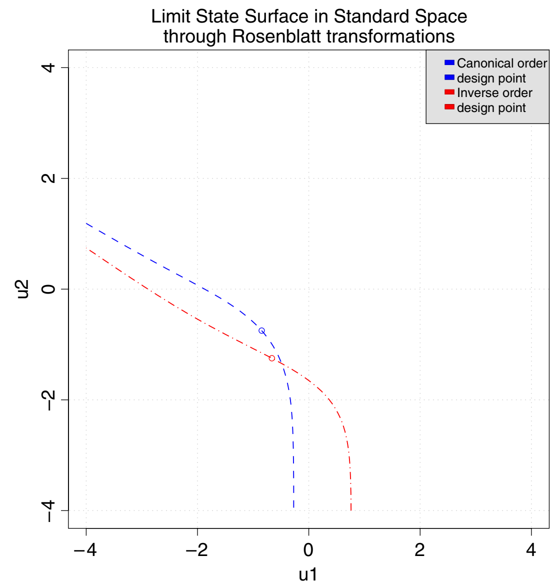


**Fig. 3.** Transformations of the limit state surface into the standard space when using the canonical order in the Rosenblatt transformation and its inverse. The copula is a Frank one with $\theta = 10$, $X_1 \sim \mathcal{E}xp(1)$ and $X_2 \sim \mathcal{E}xp(3)$. The limit state surface is $8X_1 + 2X_2 - 1 = 0$.

With the same considerations, the Rosenblatt transformation with the inverse order on the conditioning step defines the normal random vector $\underline{\tilde{U}}$ as:

$$\begin{cases} \tilde{U}_1 = \Phi^{-1} \circ F^2(X_2) \\ \tilde{U}_2 = \dfrac{\Phi^{-1} \circ F^1(X_1) - \rho \Phi^{-1} \circ F^2(X_2)}{\sqrt{1-\rho^2}} \end{cases} \tag{51}$$

which leads, in the standard space, to the other expression of the limit state surface:

$$\begin{cases} \tilde{u}_1 = \Phi^{-1} \circ F^2 \left( \dfrac{1-8\xi}{2} \right) \\ \tilde{u}_2 = \dfrac{\Phi^{-1} \circ F^1(\xi) - \rho \Phi^{-1} \circ F^2 \left( \frac{1-8\xi}{2} \right)}{\sqrt{1-\rho^2}}. \end{cases} \tag{52}$$

Fig. 2 draws the graph of the limit state surface in the standard space after both Rosenblatt transformations.

Thanks to relation (36), we can define the orthogonal matrix $\underline{\underline{Q}}$. The permutation matrix is $\underline{\underline{P}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which leads to $\underline{\underline{R}}_2 = \underline{\underline{R}}$. Furthermore, we have

$$\underline{\underline{\Gamma}} = \underline{\underline{\Gamma}}_2 = \begin{pmatrix} 1 & 0 \\ \dfrac{-\rho}{\sqrt{1-\rho^2}} & \dfrac{1}{\sqrt{1-\rho^2}} \end{pmatrix}$$

and finally

$$\underline{\underline{Q}} = \begin{pmatrix} \dfrac{\rho}{\sqrt{1-\rho^2}} & \sqrt{1-\rho^2} \\ & -\rho \end{pmatrix}.$$

We can easily verify that $\underline{\tilde{U}} = \underline{\underline{Q}}\,\underline{U}$. Furthermore, $\underline{\underline{Q}}$ is a permutation matrix with $\det(\underline{\underline{Q}}) = -1$, as the matrix $\underline{\underline{P}}$.

The director vector of the symmetry axis is $\left( \sqrt{\frac{1+\rho}{2}}, \sqrt{\frac{1-\rho}{2}} \right)$. In the numerical application drawn in Fig. 2, the symmetry axis is $(\sqrt{3}/2, 1/2)$.

The Hasofer reliability index is $\beta = 1.30$ and the FORM approximation of the event probability $p = \mathbb{P}(8X_1 + 2X_2 - 1 < 0)$ is:

$$p_{FORM} = \Phi^{-1}(-\beta) = 9.76 \times 10^{-2}. \tag{53}$$

An analytical computation of $p$ leads to the numerical result:

$$p = 8.73 \times 10^{-2} \tag{54}$$

Let us verify now the equivalence between the Nataf transformation and the Rosenblatt one (given that we consider the canonical order). The Nataf transformation leads to the normal random vector $\underline{U}$ defined as:

$$\underline{U} = \underline{\underline{\Gamma}} \begin{pmatrix} \Phi^{-1} \circ F^1(X_1) \\ \Phi^{-1} \circ F^2(X_2) \end{pmatrix} \tag{55}$$

As $\underline{\underline{\Gamma}} = \begin{pmatrix} 1 & 0 \\ -\dfrac{\rho}{\sqrt{1-\rho^2}} & \dfrac{1}{\sqrt{1-\rho^2}} \end{pmatrix}$, we have:

$$\begin{cases} U_1 = \Phi^{-1} \circ F^1(X_1) \\ U_2 = -\dfrac{\rho \Phi^{-1} \circ F^1(X_1)}{\sqrt{1-\rho^2}} + \dfrac{\Phi^{-1} \circ F^2(X_2)}{\sqrt{1-\rho^2}} \end{cases} \tag{56}$$

which is identical to the expression defined in (49).

### 6.2. Application 2: Frank copula

We consider here the Frank copula, which is a non-elliptical copula. This example proves that both limit state surfaces in the standard space associated to two different orders in the conditioning step of the Rosenblatt transformation are not linked by an orthogonal transformation. We also illustrate that, according to this conditioning order, both reliability index are different which leads to different FORM approximations of the probability.

Fig. 3 draws the graph of the limit state function in the standard space after both Rosenblatt transformations.

The respective reliability index are different in both cases:

$$\begin{cases} \beta_{CanOrd} = 1.24 \\ \beta_{InvOrd} = 1.17 \end{cases} \tag{57}$$

which leads to different FORM approximations of the event probability:

$$\begin{cases} P_{CanOrd}^{FORM} = 1.07 \times 10^{-1} \\ P_{InvOrd}^{FORM} = 1.22 \times 10^{-1}. \end{cases} \quad (58)$$

There is a difference of 14% between the two approximations, only due to the conditioning order, whereas the exact probability value is the same.

An analytical computation of $p$ leads to the numerical value:

$$p = 1.038 \times 10^{-1}. \quad (59)$$

## 7. Conclusion

This article is the third part of global reflections on isoprobabilistic transformations.

Its first main objective was to compare the generalised Nataf transformation with the Rosenblatt transformation and show that in the normal copula case, both transformations are identical.

In the use of the Rosenblatt transformation, there is a degree of freedom in the ordering of the conditioning step. This point is often presented as a drawback of this transformation, as it leads to different numerical results for the FORM and SORM approximation. The second main objective of the article was to show that, although the conditioning order has such an impact in general, in the normal copula case there is indeed no impact on the FORM and SORM approximations as well as on the reliability index. The only impact is on the importance factors in the case of correlated components for $\underline{X}$, which underlines the difficulty in interpreting such factors in the correlated case.

The Nataf transformation has been successfully generalised to produce more general standard spaces than the normal one. We showed that the Rosenblatt transformation cannot be generalised this way. Thus, for the case of a non-normal elliptical copula, one can choose between both isoprobabilistic transformations: the Rosenblatt transformation or the generalised Nataf transformation.

We illustrated these results through two numerical applications, showing the equivalence of both transformations in the normal copula case and the effect of the conditioning order in a normal and non-normal copula case.

Let us recall that the exact value of the event probability remains the same whatever the transformation we use, and whatever the conditioning order we use for the Rosenblatt transformation. It is only the FORM and SORM approximations that are potentially modified.

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